NWC TP 7030

Optimal Control Law Phenomena in Certain Adaptive Second-Order Observation Systems

by
Warren W. Willman
Applied Mathematics Research Group
Research Department

AUGUST 1989

NAVAL WEAPONS CENTER
CHINA LAKE, CA 93555-6001

Naval Weapons Center

FOREWORD

This report documents a formal analysis of asymptotic approximations to solutions of certain nonlinear optimal control problems with applications to adaptive missile autopilots. The work was sponsored by the Independent Research Program of the Naval Weapons Center and performed at the Naval Weapons Center during 1988 and 1989.

Approved by
R. L. DERR, *Head*
Research Department
1 August 1989

Under authority of
J. A. BURT
Capt., U.S. Navy
Commander

Released for publication by
G. R. SCHIEFER
Technical Director

NWC Technical Publication 7030

Published by .... Research Department
Collation .... Cover, 22 leaves
First printing .... 75 copies

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT A statement; public release; distribution unlimited.
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S) NWC TP 7030	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION Naval Weapons Center; 6b OFFICE SYMBOL (If applicable)	7a NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) China Lake, CA 93555-6001	7b. ADDRESS (City, State, and ZIP Code)
8a NAME OF FUNDING/SPONSORING ORGANIZATION Naval Weapons Center; 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code) China Lake, CA 93555-6001	10 SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO	WORK UNIT ACCESSION NO
61152N		RROONW	13807005

11 TITLE (Include Security Classification)

OPTIMAL CONTROL LAW PHENOMENA IN CERTAIN ADAPTIVE SECOND-ORDER OBSERVATION SYSTEMS (U)

12 PERSONAL AUTHOR(S)
Willman, Warren W.

13a TYPE OF REPORT	13b TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
FINAL	FROM 88 Oct TO 89 May	1989 August 1	41

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Adaptive Control, Perturbation Control, Optimal Control, Nonlinear Control, Adaptive Autopilot, Stochastic Control, Asymptotic Approximation, Missile Autopilot

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

(U) By augmenting the state vector, a certain type of adaptive optimal control problem, which is of the standard multivariate "linear-quadratic-Gaussian" form except for a relatively small degree of uncertainty in some of its parameters, is formulated as a nonlinear stochastic optimal control problem with known parameters, some of which are small. In this type of problem, the controller has noisy measurements of state components whose second, but not first, time-derivatives can be affected by the control or plant noise, and the measurement noise is small in a certain relative sense. A perturbation analysis of the conditional state covariance matrix is used to derive an asymptotic approximation of the optimal control law and the applicability of the results is demonstrated for a missile pitch autopilot.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION Uclassified	
22a NAME OF RESPONSIBLE INDIVIDUAL Warren W. Willman	22b TELEPHONE (Include Area Code) (619) 939-3457	22c OFFICE SYMBOL 3807

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

Government Printing Office: 1986—607-044

CONTENTS

INTRODUCTION AND SUMMARY

Asymptotic approximations of optimal control laws are determined here for a class of multivariate dynamic systems in which the controller has only noisy measurements of system state components whose second time-derivatives, but not first, can be directly affected by plant noise or the control. The control optimization problem for these cases would have the standard linear-quadratic-Gaussian form except for certain small nonlinearities involving slowly varying parameters, which are treated as components of an argumented state vector. Also, the measurement noise is small in a certain relative sense, which gives this control problem special properties.

A special case of this problem, which arises in homing missile guidance, was treated in Reference 1. The only nonlinearity in that case is a term in the state measurement equation that is bilinear in the parameter and control (both scalars) and gives rise to a rapidly varying term in the optimal control law. This rapidly varying term is generated as the output of a critically damped second-order system driven by a Kalman filter innovation variable.

The methods used in Reference 1 depended on special features of the case treated there; however, the same basic approach can be applied here with some modification. The result in this more general case is that—to the level of accuracy retained in the asymptotic approximations—the same sort of rapidly varying term appears in the optimal control law. This term results from bilinear measurement terms in the control and parameter variables, but not from other nonlinearities considered here. In general, this extra control term is a linear function of the output of a multivariate linear system driven by a Kalman filter innovation variable. These results are applied to the design of an adaptive pitch autopilot for a missile as a numerical example.

NOTATION

Unless otherwise stated, lower case letters denote (real finite-dimensional) column vectors and scalars. Matrices are denoted by capital Roman letters. A^T denotes the transpose of a matrix A, and tr(A) its trace if A is square.

It will be convenient to make use of three-way matrices, which are always denoted by capital Greek letters here. For continuity of notation, the following definitions are adopted for such a three-way matrix Γ, with vector x and matrices A and B of compatible dimensions, and with repeated indices denoting summation:

$(\Gamma x)_{ij} = \Gamma_{ij\sigma} x_\sigma$	(matrix)
$(Ax^T)_{ijk} = A_{ij} x_k$	(three-way matrix)
$(A\Gamma)_{ijk} = A_{i\sigma} \Gamma_{\sigma jk}$	(three-way matrix)
$(\Gamma B)_{ijk} = \Gamma_{ij\sigma} B_{\sigma k}$	(three-way matrix)
$(\Gamma')_{ijk} = \Gamma_{jki}$ and $(\Gamma^T)_{ijk} = \Gamma_{kji}$	(three-way matrices)
$[\mathrm{tr}(\Gamma)]_i = \Gamma_{\sigma i \sigma}$	(column vector, when applicable)

With these definitions, the expression $A\Gamma\Gamma B D x x^T$ is fully associative. Many other consequences are obvious. Some useful but less obvious properties are

$$\mathrm{tr}(\Gamma' x) = [\mathrm{tr}(\Gamma)]^T x$$

$$A\ \mathrm{tr}(\Gamma') = \mathrm{tr}(A\Gamma')$$

$$\mathrm{tr}(A\Gamma) = \mathrm{tr}(\Gamma A)$$

$$(\Gamma B)' = B^T\Gamma' \text{ and } (A\Gamma)'' = \Gamma'' A^T$$

$$(A\Gamma B)^T = B^T\Gamma^T A^T$$

$$(\Gamma' x)A^T = (A\Gamma)'x \text{ and } (\Gamma'' x)B = (\Gamma B)''x.$$

Partial derivatives of a scalar s with respect to a matrix A and vectors x and y are denoted by subscripts, with the convention that

$$(s_A)_{ij} = \frac{\partial s}{\partial A_{ji}},$$

$$(s_{xy})_{ij} = \frac{\partial^2 s}{\partial x_i \partial y_j}, \quad \text{and}$$

$$(s_{Ax})_{ijk} = \frac{\partial^2 s}{\partial A_{ji}\partial x_k}.$$

PROBLEM AND BASIC APPROACH

The problem treated here involves a system with dynamics

$$\dot{x} = Fx + v \tag{1}$$

$$\dot{v} = Kx + Dv + Gu + w, \tag{2}$$

a controller of which receives the vector measurement

$$z = x + h\,\mathrm{tr}(\Gamma''\theta u^T) + n \tag{3}$$

and selects the control vector u at each time instant $t \geq 0$. The time variable t is suppressed in the notation here, and the coefficient matrices may be time-varying. θ is a constant but unknown parameter vector, w and n are zero-mean Gaussian white noise processes with respective covariance parameters Q and R/m^4, and h and m are positive scalars such that

$$h \ll \frac{1}{m} \ll 1. \tag{4}$$

A priori,

$$\begin{bmatrix} x(0) \\ v(0) \\ \theta \end{bmatrix} \text{ is a Normal } \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} P_{10} & 0 & 0 \\ 0 & P_{30} & 0 \\ 0 & 0 & L_0 \end{bmatrix} \right) \tag{5}$$

random variable independent of w and n. The objective is to find a control law that minimizes the scalar performance criterion

$$J = \frac{1}{2} E \left\{ \begin{bmatrix} x_f^T & v_f^T \end{bmatrix} \begin{bmatrix} \bar{S}_1 & \bar{S}_2 \\ \bar{S}_2^T & \bar{S}_3 \end{bmatrix} \begin{bmatrix} x_f \\ v_f \end{bmatrix} + \int_0^{t_f} \left([x^T \; v^T] \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + u^T B u \right) dt \right\}, \tag{6}$$

where E denotes prior expectation and $t_f > 0$ is some specified terminal time. As usual, a control law is defined as a decision rule that, for each t in $[0, t_f)$, specifies the current control u(t) as a function of the current measurement history $\{(z(\psi), \psi) : 0 \leq \psi < t\}$. Also, in the above, P_{10}, P_{30}, and L_0 are positive definite, B(t) and R(t) are positive definite for each $t \in [0, t_f]$, $\begin{bmatrix} \bar{S}_1 & \bar{S}_2 \\ \bar{S}_2^T & \bar{S}_3 \end{bmatrix}$ is positive-semidefinite, and $Q(t)$ and $\begin{bmatrix} A_1(t) & A_2(t) \\ A_2^T(t) & A_3(t) \end{bmatrix}$ are positive-semidefinite for each $t \in [0, t_f]$. Without loss of generality, these matrices are assumed symmetric as well.

Finding such an optimal control law is very difficult, so we only consider the problem of finding an approximation thereof that is asymptotically accurate to order $h^2m^{3/2}$ for the inequalities of Equation 4, i.e., when $1/m$ and mh are both small. What is meant by such an approximation is that the control law always generates a control value u which is the same to order $h^2m^{3/2}$ as that generated by an optimal control law, except perhaps for a set of measurement histories of negligibly small probability. The size of m and the size of $1/(mh)$ if $h \neq 0$ are considered to be large enough here that the components of F, K, D, G, Γ, P_{10}, P_{30}, L_0, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$, A_1, A_2, A_3, Q, Q^{-1}, R, R^{-1}, B, B^{-1}, and their time rates of change, if any, are always of order unity by comparison.

Also, the treatment of this problem is limited here to finding the control law associated with a cost-to-go function which has the formal appearance of satisfying the Bellman equation corresponding to Equations 1-3, 5, and 6 to order $h^2m^{3/2}$. This control law would be the desired asymptotic approximation if the equations involved in the analysis are well posed and the formally higher-order terms in them are indeed so in some appropriate sense. A mathematically precise verification of these conditions is beyond the scope of this investigation, however, so in this sense the control law obtained here is only a plausible candidate for the approximation being sought. This plausibility is enhanced, though, by the fact that the actual optimal control law is well known and rigorously justified for $h = 0$ (a standard linear-quadratic-Gaussian case) and the approximation derived here for small h converges to this control law as $h \to 0$. Nevertheless, it is still important to augment this type of formal analysis by testing the results on specific numerical examples. One such example is included here, and the theory seems to give reasonable and useful results in this case.

Even formally, the asymptotic accuracy of this control law approximation is less than that obtained for the special case examined in Reference 1, where all control terms of order h^2 were included in the approximation. As it happened, the other order-h^2 control terms had an equally important effect on the performance criterion even though they were small compared to $h^2 m^{3/2}$. For the more limited purpose of investigating the salient features of control laws that are optimal for such criteria, however, it is consistent to limit the accuracy of the control law approximations here to order $h^2 m^{3/2}$. As in the example below, the performance criterion is often used only as a device to generate, by its optimization, a control law with desired properties.

MOTION-STATE AND PARAMETER ESTIMATION

The motion state (x, v) of the dynamic system and the parameter vector θ satisfy the linear system of equations

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} F & I & 0 \\ K & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ G \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} w . \tag{7}$$

Since the initial value of the composite state (x, v, θ) has a Normal prior probability distribution and since current and past values of u are presumed known to the controller, it is a standard result (Reference 2) that the current conditional probability distribution of this composite state, given current and past values of z, is also Normal, with mean and covariance matrix given by the Kalman filter equations for Equations 3, 5, and 7. If this conditional mean and covariance matrix are partitioned in the obvious way as

$$\begin{bmatrix} \hat{x} \\ \hat{v} \\ \hat{\theta} \end{bmatrix} \text{ and } \begin{bmatrix} P_1 & P_2 & E_1 \\ P_2^T & P_3 & E_2 \\ E_1^T & E_2^T & L \end{bmatrix} ,$$

these Kalman filter equations can be expressed as

$$\dot{\hat{x}} = F\hat{x} + \hat{v} + m^4[P_1 + h(\Gamma^T E_1^T)'u]R^{-1}(z - \hat{z})\ ;\ \hat{x}(0) = 0 \tag{8}$$

$$\dot{\hat{v}} = K\hat{x} + D\hat{v} + Gu + m^4[P_2^T + h(\Gamma^T E_2^T)'u]R^{-1}(z - \hat{z})\ ;$$

$$\hat{v}(0) = 0 \tag{9}$$

$$\dot{\hat{\theta}} = m^4[E_1^T + h(\Gamma^T L)'u]R^{-1}(z - \hat{z})\ ;\ \hat{\theta}(0) = 0 \tag{10}$$

$$\dot{P}_1 = FP_1 + P_1F^T + P_2 + P_2^T - m^4[P_1 + h(\Gamma^T E_1^T)'u]R^{-1}$$

$$\times [P_1 + h(E_1\Gamma)'u];\ P_1(0) = P_{10} \tag{11}$$

$$\dot{P}_2 = FP_2 + P_2^T D^T + P_1 K^T + P_3$$

$$- m^4[P_1 + h(\Gamma^T E_1^T)'u]R^{-1}[P_2 + h(E_2\Gamma)'u];\ P_2(0) = 0 \tag{12}$$

$$\dot{P}_3 = KP_2 + P_2^T K^T + DP_3 + P_3 D^T + Q -$$

$$m^4\ [P_2^T + h(\Gamma^T E_2^T)'u]R^{-1}[P_2 + h(E_2\Gamma)'u];\ P_3(0) = P_{30} \tag{13}$$

$$\dot{E}_1 = FE_1 + E_2 - m^4[P_1 + h(\Gamma^T E_1^T)'u]R^{-1}$$

$$\times [E_1 + h(L\Gamma)'u];\ E_1(0) = 0 \tag{14}$$

$$\dot{E}_2 = KE_1 + DE_2 - m^4[P_2^T + h(\Gamma^T E_2^T)'u]R^{-1}$$

$$\times [E_1 + h(L\Gamma)'u];\ E_2(0) = 0 \tag{15}$$

$$\dot{L} = -m^4[E_1^T + h(\Gamma^T L)'u]R^{-1}[E_1 + h(L\Gamma)'u];\ L(0) = L_0\ , \tag{16}$$

where

$$\hat{z} = \hat{x} + h\,\mathrm{tr}(\Gamma''\hat{\theta}u^T). \tag{17}$$

It is also convenient to define

$$\xi = m^2(z - \hat{z}), \tag{18}$$

which is the normalized innovation process for this filter. As such, ξ can be treated as a zero-mean Gaussian white noise process with covariance parameter R in determining the statistical behavior of $\hat{x}$, $\hat{v}$, and $\hat{\theta}$ (Reference 3).

It happens that L varies more slowly than the other covariance matrix partitions. A key step that takes advantage of this is to define the nominal time functions $\bar{P}_1$, $\bar{P}_2$, and $\bar{P}_3$ for $t \geq 0$ by

$$\dot{\bar{P}}_1 = F\bar{P}_1 + \bar{P}_1F^T + \bar{P}_2 + \bar{P}_2^T - m^4\bar{P}_1R^{-1}\bar{P}_1 \; ; \; \bar{P}_1(0) = P_1(0) \tag{19}$$

$$\dot{\bar{P}}_2 = F\bar{P}_2 + \bar{P}_2^TD^T + \bar{P}_1K^T + \bar{P}_3 - m^4\bar{P}_1R^{-1}\bar{P}_2 \; ; \; \bar{P}_2(0) = P_2(0) \tag{20}$$

$$\dot{\bar{P}}_3 = K\bar{P}_2 + \bar{P}_2^TK^T + D\bar{P}_3 + \bar{P}_3D^T + Q - m^4\bar{P}_2^TR^{-1}\bar{P}_2; \; \bar{P}_3(0) = P_3(0) \tag{21}$$

and let

$$N_1 = \frac{m}{h^2}(P_1 - \bar{P}_1) - mu^T\Gamma''L\Gamma^{T'}u \tag{22}$$

$$N_2 = \frac{(P_2 - \bar{P}_2)}{h^2} \tag{23}$$

$$N_3 = \frac{1}{mh^2}(P_3 - \bar{P}_3) \tag{24}$$

$$M_1 = \frac{m}{h}[E_1 + h(L\Gamma)'u] \tag{25}$$

$$M_2 = \frac{E_2}{h}. \tag{26}$$

It follows fairly directly from known properties of conditional covariance and precision matrices for multivariate Normal distributions (Reference 4) that the conditions imposed on Q and R in the preceding section imply that

All components of $\bar{P}_1$ are of order $1/m^3$,

All components of $\bar{P}_1^{-1}$ are of order m^3,

All components of $\bar{P}_2$ are of order $1/m^2$, and

All components of $\bar{P}_3$ are of order $1/m$,

except perhaps for initial transients with durations of order $1/m$. These magnitudes are established by considering the estimation problem for $\Gamma = 0$ and its usual dual for the precision matrix, and, for each i, deleting all measurements except z_i in bounding the variances of x_i and v_i (and likewise in the dual problem).

APPROXIMATE ESTIMATOR BEHAVIOR FOR A CLASS OF CONTROL LAWS

If h were zero, it is a standard result that the optimal control law for Equations 1-6 would be of the form $u = -\bar{H}\hat{x} - \bar{W}\hat{v}$, where $\bar{H}(t)$ and $\bar{W}(t)$ are certain deterministic time functions such that $\bar{H}$, $\bar{W}$, $\dot{\bar{H}}$, and $\dot{\bar{W}}$ are all of order unity. Since we are only concerned with small h here, we consider control laws of the form

$$u = -H\hat{x} - W\hat{v} + \eta \tag{27}$$

for which H and W are deterministic time functions, to be chosen for convenience later, such that H, W, $\dot{H}$, and $\dot{W}$ are of order unity, and for which the components of η are small compared to unity, except perhaps for a *negligibly improbable* set of realizations. For such a control law, it follows from Equations 8, 9, 16, and 18 through 27 that

$$\dot{L} = -(mh)^2 M_1^T R^{-1} M_1, \tag{28}$$

$$\begin{aligned}\dot{\hat{x}} = F\hat{x} + \hat{v} + \frac{1}{m}\Big\{&m^3\bar{P}_1 \\ &+ (mh)^2\left[N_1 - M_1\Gamma^{T'}(H\hat{x} + W\hat{v} - \eta)\right]\Big\}R^{-1}\xi,\end{aligned} \tag{29}$$

and

$$\begin{aligned}\dot{\hat{v}} = (K - GH)\hat{x} + (D - GW)\hat{v} + \Big\{&m^2\bar{P}_2^T \\ &+ (mh^2)\left[N_2 - M_2\Gamma^{T'}(H\hat{x} + W\hat{v} - \eta)\right]\Big\}R^{-1}\xi.\end{aligned} \tag{30}$$

Expressions for $\dot{M}_1$, $\dot{M}_2$, $\dot{N}_1$, $\dot{N}_2$, and $\dot{N}_3$ can also be obtained for this case by differentiating Equations 22-27 and substituting from Equations 8-16 and 18-21. These expressions are quite lengthy in their entirety; however, retaining only the terms that are needed to determine the optimal control to order $h^2m^{3/2}$ reduces them to

$$\dot{M}_1 = m\left[M_2 - m^3\bar{P}_1R^{-1}M_1 - m^2(L\Gamma)'W\bar{P}_2^TR^{-1}\xi\right], \tag{31}$$

$$\dot{M}_2 = -m^3\bar{P}_2^TR^{-1}M_1, \tag{32}$$

$$\begin{aligned}\dot{N}_1 = m(N_2 + N_2^T) - m^4\Big[&\bar{P}_1R^{-1}N_1 + N_1R^{-1}\bar{P}_1 \\ &- (M_1\Gamma R^{-1}\bar{P}_1 + \bar{P}_1R^{-1}\Gamma^TM_1^T)'(H\hat{x} + W\hat{v} - \eta)\Big],\end{aligned} \tag{33}$$

$$\begin{aligned}\dot{N}_2 = m\Big[&N_3 - m^3\bar{P}_1R^{-1}N_2 - m^2N_1R^{-1}\bar{P}_2 \\ &+ (m^3M_2\Gamma R^{-1}\bar{P}_1 + m^2\bar{P}_2^TR^{-1}\Gamma^TM_1^T)'(H\hat{x} + W\hat{v} - \eta)\Big],\end{aligned} \tag{34}$$

and

$$\begin{aligned}\dot{N}_3 = DN_3 + N_3D^T - m^3\Big[&\bar{P}_2^TR^{-1}N_2 + N_2^TR^{-1}\bar{P}_2 \\ &- (\bar{P}_2^TR^{-1}\Gamma^TM_2^T + M_2\Gamma R^{-1}\bar{P}_2)'(H\hat{x} + W\hat{v} - \eta)\Big].\end{aligned} \tag{35}$$

Establishing that these truncations are sufficiently accurate uses the orders of magnitude established earlier for $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$, and $\bar{P}_1^{-1}$ and follows a multivariate version of the corresponding analysis in Reference 1. This basically proceeds by assuming appropriate orders of magnitude for all the quantities involved and showing that no order-of-magnitude contradictions occur in any of the (untruncated) equations above or in the Bellman equation and approximate solution of the next section. It also entails analyzing Equations 29-35 as a noise-driven system to conclude by standard methods (Reference 3) that the M_1, M_2, N_1, N_2, and N_3 components are all random processes with values of order $m^{1/2}$, except perhaps for an initial time interval of order 1/m, which become approximately uncorrelated over a time interval of order 1/m. Since this and Equation 28 imply that L only changes by order $(mh)^2$ during the correlation time of M_1, it also follows from this argument that the difference (componentwise) between L and its prior expected value for such a control law is always small compared to unity (except for a set of realizations of negligible probability). The reason is that order-unity changes in an L-component behave basically as the sum of $1/(mh)^2$ independent random increments, each with mean of order m and variance of order m^2. Hence, the variance of this sum is of order $(mh)^2$, which is small compared to unity by assumption.

CONTROL OPTIMIZATION

Since H(t) and W(t) in Equation 27 are considered specified, the problem here reduces to that of finding an optimal control law for the perturbation control η to which we seek only an asymptotic approximation. A convenient choice of H and W will be used for this purpose, but one for which H, W, $\dot{H}$, and $\dot{W}$ are of order unity.

An optimal expected cost-to-go function can be defined consistently in terms of time and the conditional distribution of x, v, and θ (Reference 5). Thus, the Principle of Optimality of dynamic programming can be applied in the usual way (Reference 6) to derive a Bellman equation for this function, the solution of which specifies the optimal control law for η. Since the conditional distribution here is Normal and therefore specified by its first and second moments, such a solution can be expressed in terms of t, $\hat{x}$, $\hat{v}$, $\hat{\theta}$, M_1, M_2, N_1, N_2, N_3, and L. The derivation of the Bellman equation for this class of cost functions requires the conditional expected values of increments $\Delta\hat{x}$, $\Delta\hat{v}$, $\Delta\hat{\theta}$, ΔL, ΔM_1, ΔM_2, ΔN_1, ΔN_2, ΔN_3, and of quadratic products of their components, over an infinitesimal time increment Δt, given the data up to the beginning of this time increment. Since this conditioning is equivalent to conditioning on the conditional distribution of x, v, and θ at that time, these expectations can be evaluated from (the untruncated versions of)

Equations 28-35 and the corresponding equation for $\dot{\hat{\theta}}$ (which will not be needed for the level of accuracy retained below). Here, $\Delta\hat{x}$ is taken as $\dot{\hat{x}}\Delta t$, etc.

In so doing, we retain only terms up to order $h^2 m^{3/2}$ and $h^2 m^{3/2}\gamma$ (γ any product of η-components) in the resulting Bellman equation. Also, we restrict consideration to possible solutions (also denoted J) of the form

$$J = \frac{1}{2}\left[\hat{x}^T \hat{v}^T\right]\begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} + h^2\Big\{\mathrm{tr}[Q_1 N_1 + Q_2 N_2$$

$$+ \left(Q_3 + \frac{m}{2} S_3\right) N_3] + \left[\hat{x}^T \hat{v}^T\right] \mathrm{tr}\,(\Lambda[M_1 \vdots M_2])\Big\} + f(t), \quad (36)$$

where the S, Q, and Λ components are all of order unity and functions of t only, with S_1, S_3, Q_1, and Q_3 symmetric, and for which $\dot{\eta}$, the time-derivative of the associated optimal perturbation control, contributes only terms small compared to $h^2 m^{3/2}$ to the Bellman equation for the choices

$$\left.\begin{aligned} H &= B^{-1} G^T S_2^T \\ W &= B^{-1} G^T S_3 \end{aligned}\right\}. \quad (37)$$

These restrictions and choices of H and W reduce the resulting Bellman equation and boundary condition to

$$
\begin{aligned}
-J_t = \min_{\eta} \underset{/\text{data }(t)}{E} \Bigg\{ & \frac{1}{2}[x^T v^T] \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \\
& + \frac{1}{2}\eta^T B \eta + \frac{1}{2}(\hat{x}^T S_2 + \hat{v}^T S_3)\, G B^{-1} G^T \\
& \times (S_2^T \hat{x} + S_3 \hat{v}) - (\hat{x}^T S_2 + \hat{v}^T S_3)\, G B^{-1} \eta \\
& + J_{\hat{x}} \dot{\hat{x}} + J_{\hat{v}} \dot{\hat{v}} + \mathrm{tr}(J_{N_1} \dot{N}_1 + J_{N_2} \dot{N}_2 + J_{N_3} \dot{N}_3) \\
& + \mathrm{tr}\left(\frac{1}{2} J_{\hat{x}\hat{x}} \dot{\hat{x}} \dot{\hat{x}}^T + J_{\hat{x}\hat{v}} \dot{\hat{v}} \dot{\hat{x}}^T + \frac{1}{2} J_{\hat{v}\hat{v}} \dot{\hat{v}} \dot{\hat{v}}^T \right) dt \\
& + \dot{M}_1 \mathrm{tr}(J_{M_1 \hat{x}} \dot{\hat{x}} + J_{M_1 \hat{v}} \dot{\hat{v}}) dt \Bigg\} ;
\end{aligned}
$$

$$
J(t_f, \cdots) = \underset{/\text{data }(t_f)}{E} \left\{ [x^T(t_f) \vdots v^T(t_f)] \begin{bmatrix} \bar{S}_1 & \bar{S}_2 \\ \bar{S}_2^T & \bar{S}_3 \end{bmatrix} \begin{bmatrix} x(t_f) \\ v(t_f) \end{bmatrix} \right\} \tag{38}
$$

for $\dot{M}_2$ as approximated by Equation 32, where subscripts now denote partial differentiation.

From Equation 36, the indicated partial derivatives are

$$
\begin{aligned}
J_t = \frac{1}{2}\hat{x}^T \dot{S}_1 \hat{x} + \hat{x}^T \dot{S}_2 \hat{v} + \frac{1}{2}\hat{v}^T \dot{S}_3 \hat{v} + h^2\, \mathrm{tr}\Big[& \dot{Q}_1 N_1 + \dot{Q}_2 N_2 \\
& + \left(\dot{Q}_3 + \frac{1}{2} m \dot{S}_3 \right) N_3 \Big] + h^2 \left[\hat{x}^T \hat{v}^T \right] \mathrm{tr}\,(\dot{\Lambda}[M_1 \vdots M_2]) + \dot{f}
\end{aligned}
$$

$$
J_{\hat{x}} = \hat{x}^T S_1 + \hat{v}^T S_2^T + h^2\, \mathrm{tr}^T\,(\Lambda_1 [M_1 \vdots M_2])
$$

$$
J_{\hat{v}} = \hat{x}^T S_2 + \hat{v}^T S_3 + h^2\, \mathrm{tr}^T\,(\Lambda_2 [M_1 \vdots M_2])
$$

$$J_{N_1} = h^2 Q_1$$

$$J_{N_2} = h^2 Q_2$$

$$J_{N_3} = h^2 \left(Q_3 + \frac{m}{2} S_3 \right)$$

$$J_{\hat{x}\hat{x}} = S_1$$

$$J_{\hat{x}\hat{v}} = S_2$$

$$J_{\hat{v}\hat{v}} = S_3$$

$$J_{M_1\hat{x}} = h^2 \Lambda_3$$

$$J_{M_1\hat{v}} = h^2 \Lambda_4 \ ,$$

where Λ_1, Λ_2, Λ_3, and Λ_4 have components which are either zero or components of Λ. Substituting these expressions for the partial derivatives in Equation 38 and using the fact that L is approximately a deterministic time function, which from Equations 28 and 31 is independent of the perturbation control, allows the conditional expectation in Equation 38 to be evaluated to the desired accuracy with Equations 28-35 to give the minimand of Equation 38 as

$$\frac{1}{2}\{\hat{x}^T(A_1 + S_2GB^{-1}G^TS_2^T)\hat{x} + \hat{v}^T(A_3 + S_3GB^{-1}G^TS_3)\hat{v} + \eta^TB\eta$$

$$+ mh^2 \operatorname{tr}[A_3N_3 + S_3(DN_3 + N_3D^T)]\}$$

$$+ \hat{x}^T(A_2 + S_2GB^{-1}G^TS_3)\hat{v} + (\hat{x}^TS_1 + \hat{v}^TS_2)(F\hat{x} + \hat{v})$$

$$+ (\hat{x}^TS_2 + \hat{v}^TS_3)[(K - GB^{-1}G^TS_2^T)\hat{x} + (D - GB^{-1}G^TS_3)\hat{v}]$$

$$+ mh^2 \operatorname{tr}\{Q_1[N_2 + N_2^T - (m^3\bar{P}_1)R^{-1}N_1 - N_1R^{-1}(m^3\bar{P}_1)$$

$$+ m^3(M_1\Gamma R^{-1}\bar{P}_1 + \bar{P}_1R^{-1}\Gamma^TM_1^T)'$$

$$\times (B^{-1}G^TS_2^T\hat{x} + B^{-1}G^TS_3\hat{v} - \eta)]\}$$

$$+ mh^2 \operatorname{tr}[Q_2(N_3 - m^3\bar{P}_1R^{-1}N_2 - m^2N_1R^{-1}\bar{P}_2)]$$

$$+ mh^2 \operatorname{tr}\{Q_2(m^3M_2\Gamma R^{-1}\bar{P}_1 + m^2\bar{P}_2^TR^{-1}\Gamma^TM_1^T)'$$

$$\times [B^{-1}G^T(S_2^T\hat{x} + S_3\hat{v}) - \eta]\} + mh^2 \operatorname{tr}[m^2S_2\bar{P}_2^TR^{-1}N_1$$

$$+ m^3N_2^TR^{-1}\bar{P}_1S_2 - m^2Q_3(\bar{P}_2^TR^{-1}N_2 + N_2^TR^{-1}\bar{P}_2)]$$

$$+ mh^2 \operatorname{tr}\{m^2Q_3(\bar{P}_2^TR^{-1}\Gamma^TM_2^T + M_2\Gamma R^{-1}\bar{P}_2)'$$

$$\times [B^{-1}G^T(S_2^T\hat{x} + S_3\hat{v}) - \eta]\} - mh^2 \operatorname{tr}\{(m^3S_2^T\bar{P}_1R^{-1}\Gamma^TM_2^T$$

$$+ m^2M_1\Gamma R^{-1}\bar{P}_2S_2^T)'[B^{-1}G^T(S_2^T\hat{x} + S_3\hat{v}) - \eta]\} + f(t).$$

Equating the η-derivative of this expression to zero gives the minimizing perturbation control as

$$\eta = mh^2B^{-1}\mathrm{tr}\,\{\Gamma R^{-1}[m^3\bar{P}_1(2Q_1M_1 + (Q_2^T - S_2)M_2)$$

$$+ m^2\bar{P}_2(2Q_3M_2 + (Q_2 - S_2^T)M_1)]\}\,, \tag{39}$$

which results in a negligible contribution of the η-dependent terms in the minimand to $-J_t$. Collecting the remaining terms in like powers of the *a priori* random variables, evaluating the terminal boundary expectation in Equation 38, and using Equations 22-24 and the fact that $N_1 = N_1^T$ and $N_3 = N_3^T$, show that the Bellman Equation 38 is satisfied to order $h^2m^{3/2}$ by the function J of Equation 36 if f and the S, Q, and Λ components satisfy the terminal-value system of ordinary differential equations:

$$-\dot{S}_1 = S_1F + F^TS_1 + S_2K + K^TS_2^T + A_1$$

$$- S_2GB^{-1}G^TS_2^T\,;\; S_1(t_f) = \bar{S}_1 \tag{40}$$

$$-\dot{S}_2 = S_1 + S_2D + F^TS_2 + K^TS_3 + A_2$$

$$- S_2GB^{-1}G^TS_3\,;\; S_2(t_f) = \bar{S}_2 \tag{41}$$

$$-\dot{S}_3 = S_2 + S_2^T + S_3D + D^TS_3 + A_3$$

$$- S_3GB^{-1}G^TS_3\,;\; S_3(t_f) = \bar{S}_3 \tag{42}$$

$$-\dot{Q}_1 = m\left\{R^{-1}\left[\frac{1}{2}(m^2\bar{P}_2)\,(S_2^T - Q_2) - (m^3\bar{P}_1)Q_1\right]\right.$$

$$\left.+\left[\frac{1}{2}(S_2 - Q_2^T)\,(m^2\bar{P}_2)^T - Q_1(m^3\bar{P}_1)\right]R^{-1}\right\};$$

$$Q_1(t_f) = 0 \tag{43}$$

$$-\dot{Q}_2 = m\left[(S_2^T - Q_2)(m^3\bar{P}_1)R^{-1} + 2Q_2 - 2Q_3(m^2\bar{P}_2)^T R^{-1}\right];$$

$$Q_2(t_f) = 0 \tag{44}$$

$$-\dot{Q}_3 = \frac{m}{2}(Q_2 - S_2^T + Q_2^T - S_2 + S_3GB^{-1}G^TS_3)\ ;\ Q_3(t_f) = 0 \tag{45}$$

$$-\dot{\Lambda}_i = \text{similar expression};\ \Lambda_i(t_f) = 0;\ i = 1, \ldots, 4$$

$$-\dot{f} = \text{similar expression};\ f(t_f) = \frac{1}{2}\ \mathrm{tr}\left[\bar{S}_1\bar{P}_1(t_f)\right.$$

$$\left. + 2\bar{S}_2\bar{P}_2^T(t_f) + \bar{S}_3\bar{P}_3(t_f)\right],$$

since these S, Q, and Λ components will be of order unity. As a consequence of the dynamic programming procedure (Reference 6), the optimal perturbation control for Equations 27 and 37 is then given to order $h^2m^{3/2}$ by the corresponding η of Equation 39.

It also follows from differentiating Equation 39, from substituting for the derivatives in the resulting expression, and from the previously established orders of magnitude for the quantities involved that the time-derivative of the η is small enough that it would contribute only negligibly to the Bellman equation, as was assumed.

IMPLEMENTATION AND EXTENSION

Defining the matrices

$$C_1 = m^3R^{-1}\bar{P}_1,$$

$$C_2 = m^2R^{-1}\bar{P}_2,$$

$$Y = \frac{1}{4}S_3GB^{-1}G^TS_3,$$

$$S = \frac{1}{4}(S_2^T - Q_2) + \frac{1}{4}(S_2 - Q_2^T) \qquad (= S^T),$$

and

$$A = \frac{1}{4}(S_2^T - Q_2) - \frac{1}{4}(S_2 - Q_2^T) \qquad (= -A^T)$$

and using Equations 43-45 gives

$$-\dot{Q}_1/m = C_2(S + A) + (S + A)C_2^T - C_1Q_1 - Q_1C_1^T,$$

$$-\dot{Q}_3/m = 2(Y - S),$$

$$-\dot{S}/m = \frac{1}{2}(Q_3C_2^T - SC_1^T - Q_1) + \frac{1}{2}(C_2Q_3$$

$$- C_1S - Q_1) + \frac{1}{2}(C_1A - AC_1^T),$$

and

$$-\dot{A}/m = -\frac{1}{2}(AC_1^T + C_1A) + \frac{1}{2}(Q_3C_2^T - SC_1^T - C_2Q_3 + C_1S).$$

Since C_1, C_2, Y, and C_1^{-1} are all of order unity, this system of differential equations settles in reverse time with time constants of order 1/m to its steady-state solution with

$$A = 0,$$

$$S = Y, \tag{46}$$

$$C_1Q_1 + Q_1C_1^T = C_2Y + YC_2^T, \tag{47}$$

and

$$Q_1 = Q_1^T = \frac{1}{2}(C_2Q_3 + Q_3C_2^T - C_1Y - YC_1^T). \quad (48)$$

By the preceding definitions and Equation 39,

$$\eta = 2mh^2B^{-1} \operatorname{tr} [\Gamma(C_1Q_1 - C_2S)M_1 + \Gamma(C_2Q_3 - C_1S)M_2].$$

From Equations 46-48, therefore,

$$\eta = 2mh^2B^{-1} \operatorname{tr} (\Gamma C_1^{-1}C_2YM_2)$$

to order $h^2m^{3/2}$, except within order 1/m of the terminal time. From the definitions of C_1, C_2, and Y, this perturbation control is

$$\eta = \frac{1}{2}h^2B^{-1}(\Gamma\bar{P}_1^{-1}\bar{P}_2S_3GB^{-1}G^TS_3M_2). \quad (49)$$

The final result can be summarized more conveniently by absorbing h and m into Γ and R, so that

$$z = x + \operatorname{tr} (\Gamma''\theta u^T) + GWN(R). \quad (50)$$

Except within a terminal time interval of order 1/m (which will no longer be considered here), the optimal control law can then be approximated to order $h^2m^{3/2}$ as

$$u = -B^{-1}G^T(S_2\hat{x} + S_3\hat{v})$$

$$+ \frac{1}{2}B^{-1} \operatorname{tr} (\Gamma P_1^{-1}P_2S_3GB^{-1}G^TS_3M_2), \quad (51)$$

where S_1, S_2, and S_3 are as determined by Equations 40-42 and $\hat{x}$, $\hat{v}$, P_1, P_2, and M_2 are generated in real time from the incoming measurements z by the differential equation system

$$\dot{\hat{x}} = F\hat{x} + \hat{v} + (P_1 + E_1\Gamma^{T'}u)R^{-1}(z - \hat{z})\ ;\ \hat{x}\,(0)\ \text{given}$$

$$\dot{\hat{v}} = K\hat{x} + D\hat{v} + Gu + (P_2^T + E_2\Gamma^{T'}u)R^{-1}(z - \hat{z})\ ;\ \hat{v}\,(0)\ \text{given}$$

$$\dot{\hat{\theta}} = (E_1^T + L\Gamma^{T'}u)R^{-1}(z - \hat{z})\ ;\ \hat{\theta}\,(0) = 0$$

$$\dot{P}_1 = FP_1 + P_1F^T + P_2 + P_2^T - (P_1 + E_1\Gamma^{T'}u)R^{-1} \times [P_1 + (E_1\Gamma)'u]\ ;\ P_1(0) = P_{10}$$

$$\dot{P}_2 = P_1K^T + FP_2 + P_2^TD^T + P_3 - (P_1 + E_1\Gamma^{T'}u)R^{-1} \times [P_2 + (E_2\Gamma)'u]\ ;\ P_2(0) = 0$$

$$\dot{P}_3 = KP_2 + P_2^TK^T + DP_3 + P_3D^T + Q - (P_2^T + E_2\Gamma^{T'}u)R^{-1} \times [P_2 + (E_2\Gamma)'u]\ ;\ P_3(0) = P_{30}$$

$$\dot{E}_1 = FE_1 + E_2 - (P_1 + E_1\Gamma^{T'}u)R^{-1}[E_1 + (L\Gamma)'u]\ ;\ E_1(0) = 0$$

$$\dot{E}_2 = KE_1 + DE_2 - (P_2^T + E_2\Gamma^{T'}u)R^{-1}[E_1 + (L\Gamma)'u]\ ;\ E_2(0) = 0$$

$$\dot{L} = -(E_1^T + L\Gamma^{T'}u)R^{-1}[E_1 + (L\Gamma)'u]\ ;\ L(0) = L_0$$

$$\dot{M}_1 = -P_1R^{-1}M_1 - M_2 + (L\Gamma)'B^{-1}G^TS_3P_2^TR^{-1}(z - \hat{z})\ ;\quad M_1(0) = 0 \tag{52}$$

$$\dot{M}_2 = P_2^TR^{-1}M_1\ ;\ M_2(0) = 0 \tag{53}$$

with

$$\hat{z} = \hat{x} + \mathrm{tr}\,(\Gamma''\hat{\theta}u)$$

and u as given by Equation 51.

This result can be extended to a more general case. In the preceding context, with c and e denoting the composite vectors $\begin{bmatrix} x \\ v \end{bmatrix}$ and $\begin{bmatrix} x \\ v \\ \theta \end{bmatrix}$, respectively, the dynamics in this extension are of the form

$$\left.\begin{aligned} \dot{c} &= \bar{F}c + \bar{G}u + \mathrm{tr}\,(\Delta ee^T) + \mathrm{tr}\,(\Omega eu^T) + (I + \Psi e)\begin{bmatrix} 0 \\ w \end{bmatrix} \\ \dot{\theta} &= (mh)w_2 \end{aligned}\right\}, \tag{54}$$

where $\bar{F} = \begin{bmatrix} F & I \\ K & D \end{bmatrix}$ and $\bar{G} = \begin{bmatrix} 0 \\ G \end{bmatrix}$, the state measurements are of the form

$$z = x + \mathrm{tr}\,(\Gamma\theta u^T) + \mathrm{tr}\,(\Lambda cu^T) + \mathrm{tr}\,(\Phi ee^T) + n, \tag{55}$$

and the criterion to be minimized is of the form

$$J = \frac{1}{2}\,E\left\{c_f^T(S + \Pi c_f)c_f + \int_0^{t_f}\left[a^Tu + c^T(A + \Xi e)c \right.\right.$$

$$\left.\left. + u^T(B + \Sigma e + \Theta u)u\right]dt\right\}.$$

Here, $B(t)$ is symmetric and positive-definite; S_1, $A(t)$, and the covariance parameter of w_2 are symmetric and positive-semidefinite; the components of $a(t)$, $A(t)$, $B(t)$, $B^{-1}(t)$, S, and the covariance parameter of w_2 are of order unity; all the components of the three-way matrices are of order h; and w_2 is statistically independent of w, h, and the prior distribution. This is a special case of the class of control problems treated to order-h accuracy in Reference 7 for R and R^{-1} of order unity, where R now denotes the covariance parameter of n itself (and so is of order m^{-4} here).

Adapting the derivation of Reference 7 to the more accurate measurements here and retaining any additional terms affecting the result to order $h^2 m^{3/2}$, simply add the second "dither" term of Equation 51 to the control law of Reference 7 for this case. M_2 is generated by Equations 52 and 53, where P_1 and P_2 denote the corresponding xx^T and xv^T covariance matrix partitions of the standard extended Kalman filter for Equations 54 and 55 and where

$$\hat{z} = \hat{x} + \mathrm{tr}\,(\Gamma''\hat{\theta}u^T) + \mathrm{tr}\,(\Omega''\hat{\theta}\hat{x}^T).$$

EXAMPLE - MISSILE PITCH AUTOPILOT
(All variables are scalars in this section)

The attitude dynamics of a missile in its pitch plane are often approximated as

$$\dot{\alpha} = q - g/V_m \tag{56}$$

$$\dot{q} = A\alpha + B\delta \tag{57}$$

with

$$g = F\alpha + H\delta, \tag{58}$$

where (see Figure 1)

α = angle of attack

q = pitch angle rate

g = missile acceleration (in pitch plane) normal to its body axis

V_m = missile airspeed

δ = fin deflection angle

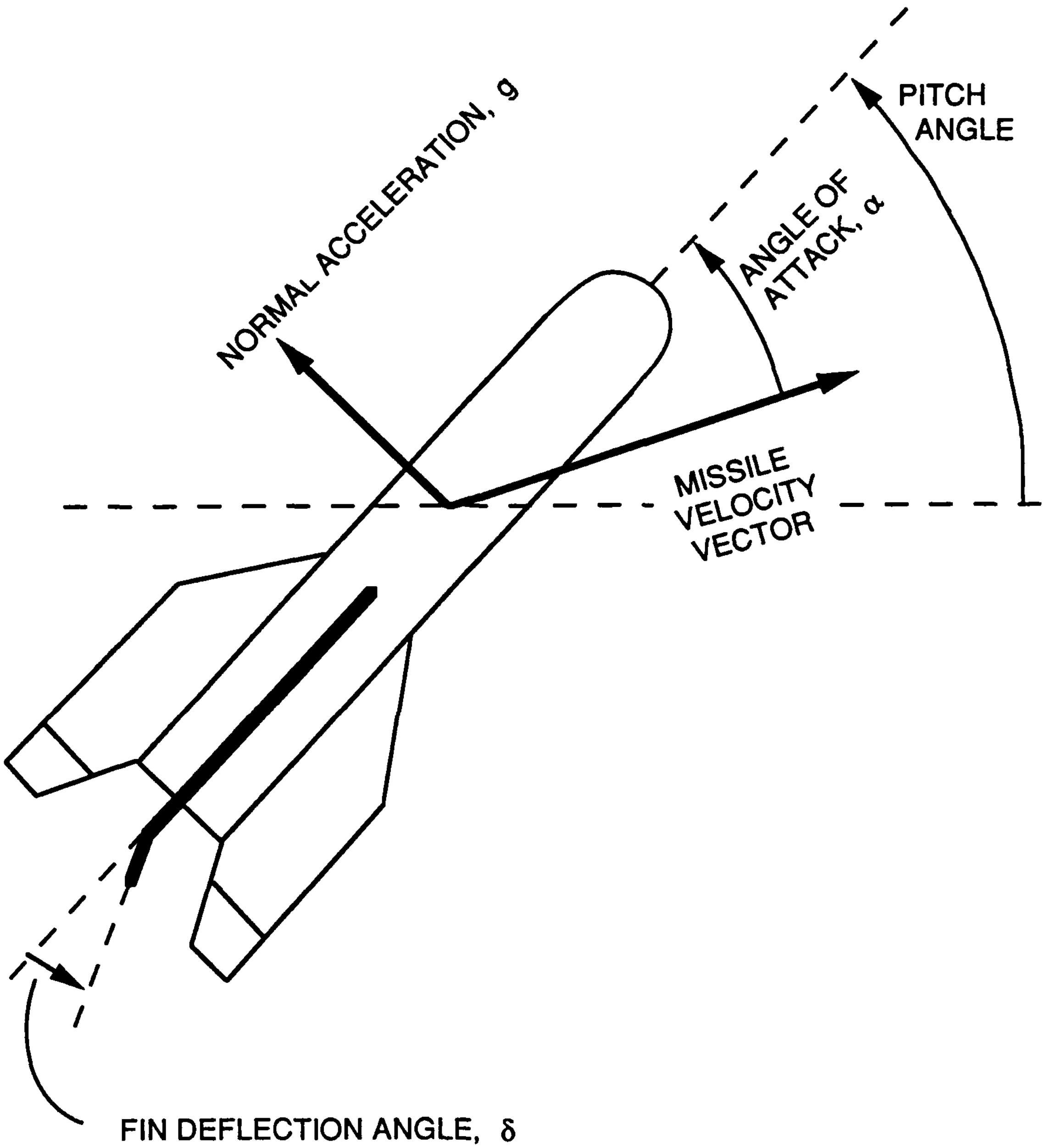


FIGURE 1. Missile Attitude in Pitch Plane.

and where

$$A = \frac{Qsd}{I} C_{m\alpha}$$

$$B = \frac{Qsd}{I} C_{m\delta}$$

$$F = \frac{Qs}{M} C_{n\alpha}$$

$$H = \frac{Qs}{M} C_{n\delta}$$

with

Q = dynamic pressure ($\frac{1}{2}V_m^2$ x atm. density)

s = missile cross-sectional area parameter

d = missile length parameter

I = missile rotational moment of inertia (in pitch plane)

M = missile mass,

and $C_{m\alpha}$, $C_{m\delta}$, $C_{n\alpha}$, and $C_{n\delta}$ are the usual "aerodynamic derivatives." These areodynamic derivatives are generally treated as constants for any given missile, although in reality they depend at least weakly on Mach number, angle of attack, and other variables. The fin deflection δ is considered the control variable here, and the controller is assumed to have measurements only of the current normal acceleration g. Measurements of the pitch rate q also could be obtained from gyroscopes, but such additional instrumentation would add to the complexity and fragility of a missile. Hence, it is of interest to see what can be done without it. The dynamic system in this formulation would be that of Equations 56 and 57, with

Equation 58 substituted for g in Equation 56 if A, B, C, D, and V_m are treated as known quantities.

The objective in designing the control law (autopilot) is to make the actual acceleration g(t) follow any reasonable commanded history c(t). Equilibrium conditions can be found for constant c by solving Equations 56-58 with $\dot{\alpha} = \dot{q} = 0$ and g = c, which gives

$$\bar{\alpha} = \left(\frac{B}{FB - AH}\right) c, \tag{59}$$

$$\bar{q} = c/V_m, \tag{60}$$

and

$$\bar{\delta} = \left(\frac{A}{AH - FB}\right) c \tag{61}$$

for the corresponding values of α, q, and δ. A simple option would be to use Equation 61 as an open-loop control law, using nominal values of A, B, F, and H. However, missiles are typically so underdamped that this does not work well even at the nominal speed and altitude to which these values correspond (see Figure 2a).

In this context, however, if one defines

$$x = \alpha - \bar{\alpha}, \tag{62}$$

$$v = q - g/V_m, \tag{63}$$

$$u = \delta - \bar{\delta}, \tag{64}$$

and the measurement variable

$$z = \frac{g - c}{F} - \frac{H}{F}(\delta - \bar{\delta}), \tag{65}$$

then it follows from Equations 56-61 that

$$\dot{x} = v, \tag{66}$$

$$\dot{v} = Ax + Bu, \tag{67}$$

and

$$z = x \tag{68}$$

for constant c at the nominal conditions. Since g is mainly the result of body lift for a missile (i.e., F » H), it is approximately proportional to α. Hence, it is reasonable to seek a control law for which x (deviation of α from its equilibrium value for the commanded acceleration c) behaves as a high-frequency critically damped sinusoid. If the full state (x, v) could be measured, the control law that minimizes the criterion

$$J = E\left[\int_{t_0}^{t_f} (x^2 + G^2u^2)\, dt\right], G > 0 \tag{69}$$

for the system of Equations 62 and 63 with white noise added to $\dot{v}$ can be found by standard methods (Reference 8). As long as $G \ll |B/A|$, this control law is approximately

$$u = -\frac{1}{B}\left(\frac{|B|}{G}\, x + \sqrt{\frac{2|B|}{G}}\, v\right) \tag{70}$$

for $t_f - t \gg \sqrt{G/|B|}$, i.e., for any fixed t as the criterion is changed so that $t_f \rightarrow \infty$. Substituting Equation 70 for u in Equation 67 shows that x behaves as a damped sinusoid with natural frequency

$$\Omega = \left[(B/G)^2 - A^2\right]^{1/4}$$

and damping ratio

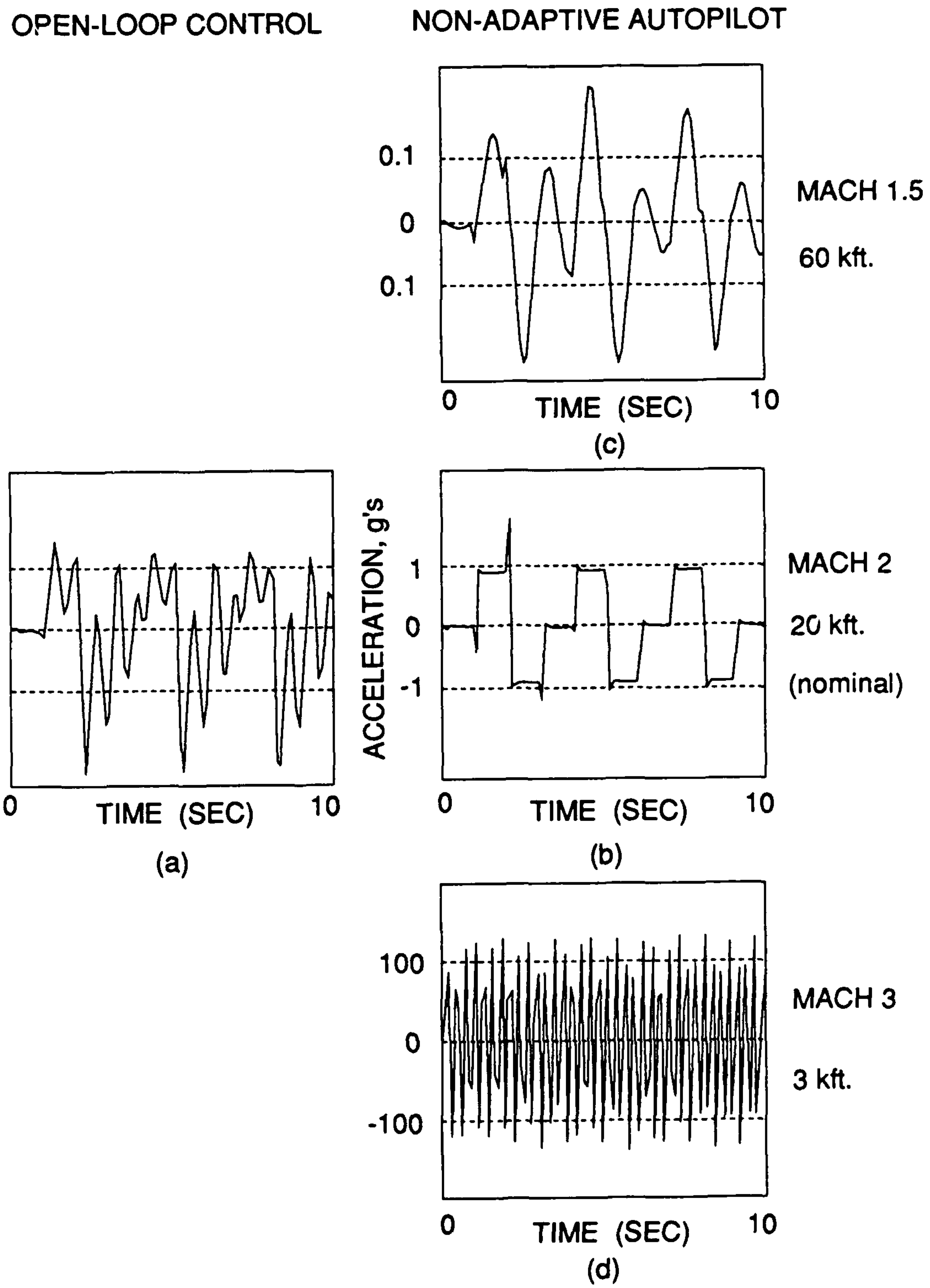


FIGURE 2. Response to 1g Step Command.

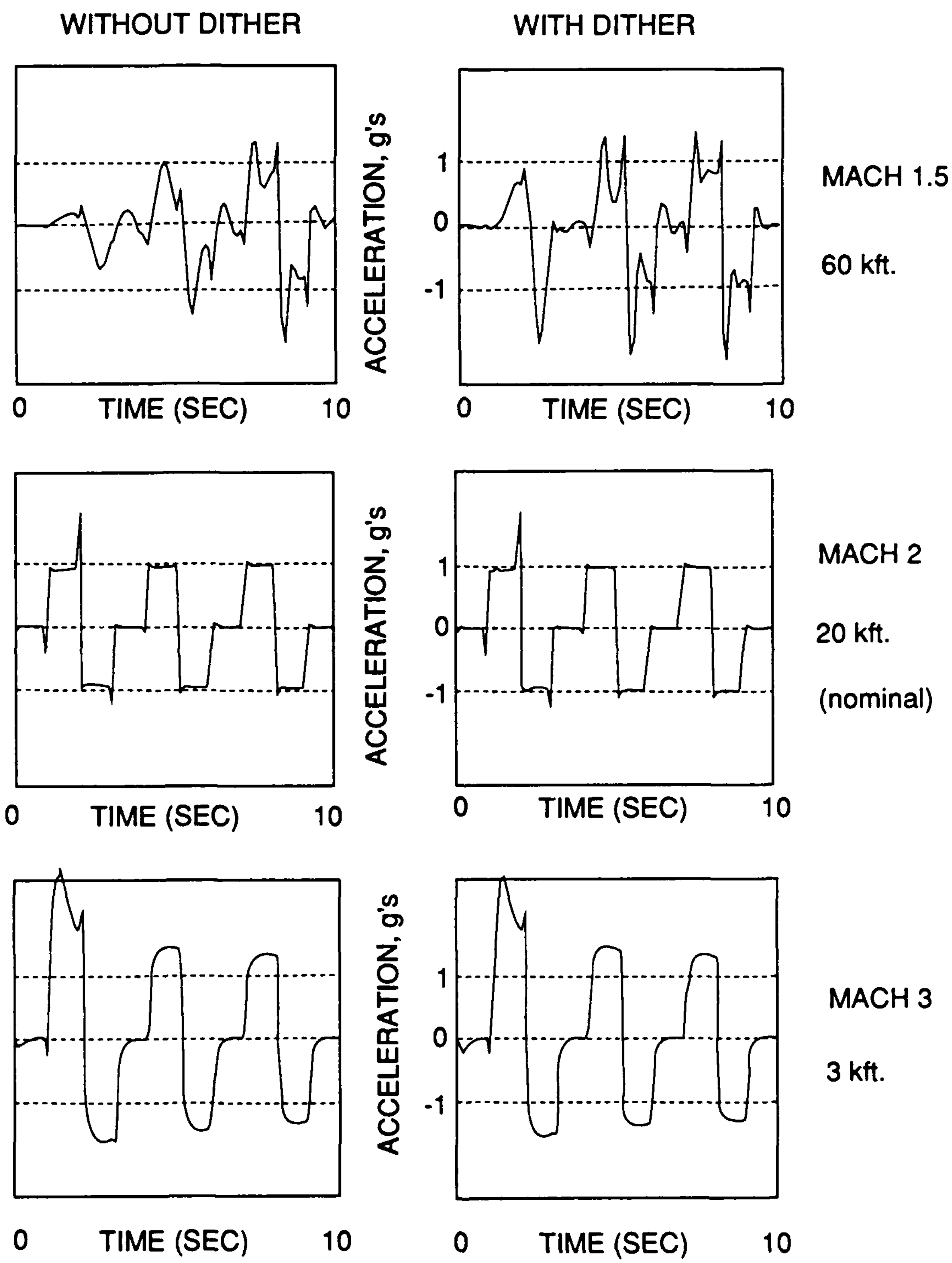


FIGURE 3. Adaptive Autopilot Response.

$$\zeta = \sqrt{\frac{1}{2} + \frac{1}{2}A/\Omega^2}.$$

For G small enough that $\Omega \gg \sqrt{|A|}$, $\zeta \cong 1/\sqrt{2}$ (critical damping) and x behaves as desired. Under these conditions, $\Omega \cong \sqrt{|B|/G}$; thus, using the control law

$$u = -(\Omega^2 x + \sqrt{2}\,\Omega v)/B \tag{71}$$

would make x behave approximately as a critically damped sinusoid of frequency Ω as long as $\Omega \gg |A|$.

Only the x-component of the state (x, v) is measured directly, however, so v must be estimated. One way of doing this is to replace x and v in Equation 71 by the estimates produced from the measurements of Equation 65 by the Kalman filter corresponding to Equations 66-68, with white noises added to Equations 66 and 68. If the respective variance parameters of these noises are q_p and r, this filter will have a settling time τ of about $(r/q_p)^{1/4}$ (see below). This settling time completely determines the effect of q_p and r on the filter estimates and should be chosen so that $\Omega\tau < 1$ for the purpose of using these estimates in Equation 71.

Finally, some of the approximation errors can be canceled out by using the postulated dynamics to replace the estimated value of x in this control law by equivalent quantities involving the actual and commanded values of the normal acceleration, which are actually known directly. From Equation 58 and the definitions of $\bar{\alpha}$ and $\bar{\delta}$,

$$g - c = F(\alpha - \bar{\alpha}) + H(\delta - \bar{\delta}).$$

But from the definitions of x and u, the control of Equation 71 is

$$\delta - \bar{\delta} = -[\Omega^2(\alpha - \bar{\alpha}) + \sqrt{2}\,\Omega\hat{v}]/B,$$

where $\hat{v}$ denotes the Kalman filter estimate of v, so it can be expressed as

$$\delta - \bar{\delta} = -\{\Omega^2[g - c - H(\delta - \bar{\delta})]/F + \sqrt{2}\,\Omega\hat{v}\}/B.$$

Solving for δ and substituting from Equation 61 for $\bar{\delta}$ then gives

$$\delta = \frac{Ac}{AH - FB} - \frac{\Omega^2(g - c) + \sqrt{2}\,F\Omega\hat{v}}{FB - \Omega^2 H} \tag{72}$$

as the desired control law, where the role of the Kalman filter is now limited to providing $\hat{v}$ from the derived measurements of Equation 65, which from Equation 61 can be expressed as

$$z = \frac{g - H\delta}{F} + \frac{Bc}{AH - FB}. \tag{73}$$

This is essentially the concept of plant inversion via state feedback described in Reference 9.

A control law of this type performs well at the nominal conditions for which it is designed (see Figure 2b), as might be expected from its use of feedback. It can still perform badly, however, if the missile speed and altitude are very different from these nominal values (see Figures 2c and 2d). This shows the need for an adaptive extension of such a control law.

It was found empirically that the dynamic pressure Q and, to a lesser extent, the aerodynamic derivative $C_{m\alpha}$ are important parameters to estimate adaptively. For this purpose, it is preferable to use ln(Q) as the dynamic-pressure parameter, since it can legitimately have a Normal distribution; also, it is preferable to use Equation 58 to eliminate α in Equation 57 so that g can be used there as a directly known quantity to cancel out additional approximation errors. (This latter stratagem did not help in the case of the simple nonadaptive autopilot.) If variations in the other aerodynamic derivatives are ignored, this allows the dynamic system to be expressed as

$$\dot{\alpha} = q - g/V_m, \tag{74}$$

$$\dot{q} = \left(\tilde{B} - \psi\tilde{A}\tilde{H}/\tilde{F}\right)e^{\theta}\delta + \frac{\tilde{A}g}{\tilde{F}}\psi + w_p, \tag{75}$$

$$\dot{\theta} = w_\theta, \tag{76}$$

and

$$\dot{\psi} = w_\psi, \tag{77}$$

where the lateral acceleration g is treated as a known quantity. $\tilde{A}$, $\tilde{B}$, $\tilde{F}$, and $\tilde{H}$ are evaluated at some nominal missile altitude and airspeed V_m. Also,

$$\theta = \ln(Q/\tilde{Q})$$

$$\psi = C_{m\alpha}/\tilde{C}_{m\alpha}$$

for

$$\left.\begin{aligned} \tilde{Q} &= Q \\ \tilde{C}_{m\alpha} &= C_{m\alpha} \end{aligned}\right\} \quad \text{at the nominal altitude and airspeed,}$$

and w_p, w_θ, and w_ψ are independent noise processes introduced for a realistic degree of uncertainty. The variance parameters used for these noises are, respectively,

q_p, left as a design parameter

q_θ = (prior variance of θ)/T

$$q_\psi = (\text{prior variance of } \psi)/T,$$

where T is the time scale over which the parameters can be expected to change drastically because of altered flight conditions.

The control optimization approach used above for the nominal flight conditions can be extended to the present situation by considering $\bar{\alpha}, \bar{q}, \bar{\delta}$, x, v, and u of Equations 59-64 for the actual but *a priori* unknown flight conditions and the variable z, defined as

$$z = \frac{g - c}{\tilde{F}} - \frac{\tilde{H}}{\tilde{F}}(\delta - \bar{\delta}), \tag{78}$$

where $\tilde{F}$ and $\tilde{H}$ are still the values of F and H at the known nominal condition. Then it follows from Equations 59-61, 74, and 75 that (for a constant commanded acceleration c),

$$\dot{x} = v, \tag{79}$$

$$\dot{v} = e^\theta \tilde{B} u + \psi \frac{\tilde{A}}{\tilde{F}}(g - c - \tilde{H} e^\theta u) + w_p, \tag{80}$$

and

$$z = e^\theta x + \frac{\tilde{H}}{\tilde{F}}(e^\theta - 1)u. \tag{81}$$

Consider choosing the autopilot design frequency Ω so that $\Omega \gg |A|$ for any "reasonable" flight condition. If z could be measured and δ computed from u, the desired autopilot behavior could then be approximated by generating u with the control law that minimizes the criterion 69, with $G = |B|/\Omega$, for the dynamics of Equations 76, 77, 79, and 80 (see discussion of Equations 69-71). Since $\bar{\delta}$ is defined in terms of the unknown actual flight conditions, the controller cannot really measure z or determine the actual control δ from u. However, $\bar{\delta}$ is a rather small quantity. As an expedient approximation, the optimal control law for u is determined as if $\bar{\delta}$ (but not A, B, F, and H) were known and then added to an estimate of $\bar{\delta}$. For this purpose, $\tilde{A}$ is also ignored in Equation 80 as relatively small, T is assumed large enough that w_θ and w_ψ in Equations 76 and 77 can be ignored, and a low-intensity noise n is added to z of Equation 81 to provide a realistic degree of uncertainty. Then, for small θ, this control problem becomes approximately that of minimizing

$$J = E\left\{\int_{t_0}^{t_f}\left[x^2 + \left(\frac{\tilde{B}^2}{\Omega^4}\right)(1 + 2\theta)u^2\right]dt\right\} \tag{82}$$

(with t_f large) for the dynamics

$$\dot{x} = v, \tag{83}$$

$$\dot{v} = \tilde{B}u + \tilde{B}\theta u + w_p, \tag{84}$$

$$\dot{\theta} = 0, \tag{85}$$

and the state measurements

$$z = x + \theta x + \frac{\tilde{H}}{\tilde{F}}\theta u + n. \tag{86}$$

The variance parameter of n in Equation 86 is taken as $\tau^4 q_p$, where τ is some specified time constant such that $\Omega\tau \ll 1$.

Since τ is small, Equations 82-86 become an optimal control problem of the form analyzed above for $\theta \to 0$. Applying the results developed there shows that (away from the terminal boundary at t_f) the optimal control law is approximated asymptotically by

$$u = -\frac{\Omega^2\hat{x} + \sqrt{2}\,\Omega\hat{v}}{\tilde{B}(1 + \hat{\theta})} + \left[\frac{\Omega}{\tilde{B}(1 + \hat{\theta})}\right]^2 \frac{\tilde{H}}{\tilde{F}}\eta\,, \tag{87}$$

where η in the second control term (called dither) is generated by

$$\left.\begin{aligned} \dot{\eta} &= \gamma/\tau \\ \dot{\gamma} &= \frac{1}{\tau}\left[\frac{2}{\tau^2}L(z - \hat{z}) - \eta - \sqrt{2}\,\gamma\right] \end{aligned}\right\}, \tag{88}$$

and where $\hat{x}$, $\hat{v}$, and $\hat{\theta}$ are the approximate conditional expectations of x, v, and θ produced by the extended Kalman filter for Equations 83-86, L is the corresponding conditional variance of θ, and $\hat{z}$ is the approximate conditional expectation of z, namely

$$\hat{z} = (1 + \hat{\theta})\hat{x} + \frac{\tilde{H}}{\tilde{F}}\hat{\theta}u. \tag{89}$$

To convert this result into a feasible control law as a sophisticated user might, the quantity $(1 + \hat{\theta})$ is replaced by $e^{\hat{\theta}}$ for robustness and the reasoning used in deriving Equation 72 from Equation 71 is applied to Equation 87 to obtain

$$\delta = \hat{\delta} - \frac{\Omega^2(g - c) + \sqrt{2}\,\hat{F}\Omega\hat{v} - (\Omega^2\hat{H}/\hat{B})\eta}{\hat{F}\hat{B} - \Omega^2\hat{H}}\,, \tag{90}$$

where

$$\hat{\delta} = \frac{\hat{A}c}{\hat{A}\hat{H} - \hat{F}\hat{B}}\,, \tag{91}$$

$$\left.\begin{aligned}\hat{A} &= \tilde{A}\hat{\psi}e^{\hat{\theta}} \\ \hat{B} &= \tilde{B}e^{\hat{\theta}} \\ \hat{F} &= \tilde{F}e^{\hat{\theta}} \\ \hat{H} &= \tilde{H}e^{\hat{\theta}} \\ \hat{v} &= \hat{q} + g/V_m\end{aligned}\right\}, \tag{92}$$

and where $\hat{q}$, $\hat{\theta}$, and $\hat{\psi}$ are the conditional mean estimates for q, θ, and ψ generated by the extended Kalman filter for the more accurate equations of motion (Equations 74-77). L in Equation 88 is likewise the conditional variance of θ from this more accurate filter. The measurement for this more accurate filter is constructed as

$$z = \frac{g - \tilde{H}\delta}{\tilde{F}}, \tag{93}$$

so that, with the same measurement noise added, it follows from Equation 58 and the definition of θ that

$$z = e^{\theta}\alpha + \frac{\tilde{H}}{\tilde{F}}(e^{\hat{\theta}} - 1)\,\delta + n. \tag{94}$$

Also, its (approximate) conditional mean is therefore

$$\hat{z} = e^{\hat{\theta}}\hat{\alpha} + \frac{\tilde{H}}{\tilde{F}}(e^{\hat{\theta}} - 1)\,\delta. \tag{95}$$

In summary, the adaptive control law derived in this way is that specified by Equations 88, 90-93 and 95 with $\hat{v}$, $\hat{\theta}$, $\hat{\psi}$, and L (which is var(θ)) as generated from the measurements of Equation 95 by the extended Kalman filter for Equations 74-77 and 94.

Figure 3 shows the performance of this control law, both with and without the dither term containing η, in a realistic missile and aerodynamic simulation. The missile's fin deflection δ was limited to ±25 degrees, so the control laws actually used saturated at those values. Three flight conditions were simulated:

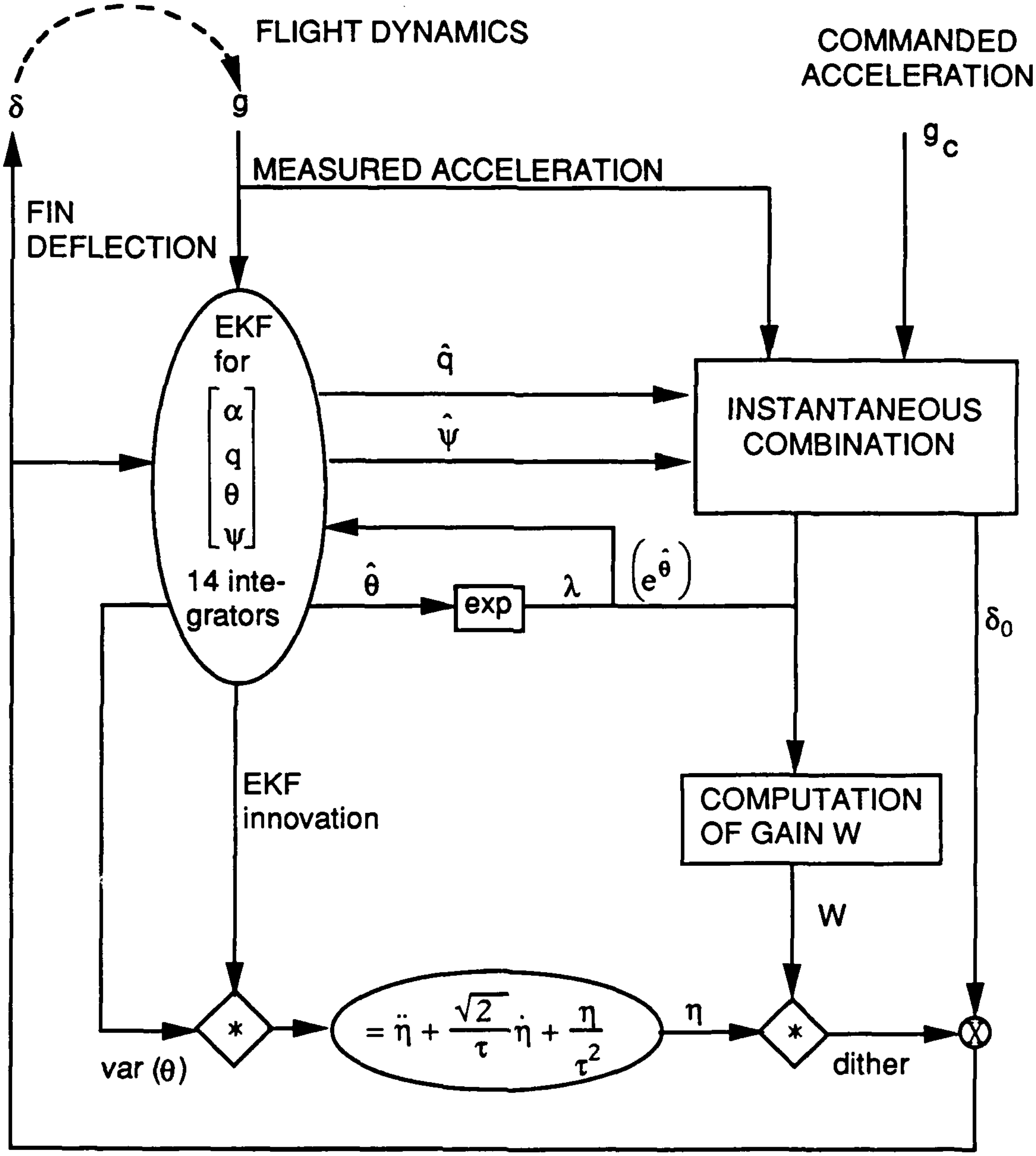
1. 3,000-ft. altitude at Mach 3
2. 20,000-ft. altitude at Mach 2 (nominal)
3. 60,000-ft. altitude at Mach 1.5.

The commanded acceleration c in each case was the step function indicated in Figure 3. These flight conditions covered a dynamic pressure variation over a factor of 65 as well as a factor of 2 variation in Mach number. For comparison, Figure 2 shows the corresponding performance of the nonadaptive version of this control law and also that of the open-loop control law for the nominal flight condition only. The nonadaptive autopilot was clearly a failure at flight conditions 1 and 3 (note the scale changes in Figures 2b and 2d), although it and the adaptive autopilot performed almost identically at the nominal condition 2. It was always helpful to use the dither control component in the adaptive autopilot, but its effect was barely noticeable except at the high-altitude flight condition 3, where the time to adapt to the nonnominal flight parameters was reduced by one-half. The dynamic pressure was so low at flight condition 3 that the simulated missile needed a 10 degree angle of attack to achieve even the 1-gravity limits of the commanded normal acceleration.

The operation of the adaptive autopilot is displayed schematically in Figure 4, where the definitions

$$\left.\begin{array}{r} \rho = C_{n\delta}/C_{n\alpha} \\ \text{and} \\ r = C_{m\delta}/C_{m\alpha} \end{array}\right\} \text{(at the nominal flight condition)}$$

are adopted for convenience.



$$W = 2\rho\Omega^2 / [\tau^4 \tilde{B}\lambda(\tilde{B}\lambda - \rho\Omega^2)]$$

$$\delta_0 = \frac{g_c/\tilde{F}\lambda}{\rho - r/\hat{\psi}} - \frac{\Omega^2(g - g_c)/\tilde{F}\lambda + \sqrt{2}\,\Omega(\hat{q} - g/V_m)}{\tilde{B}\lambda - \rho\Omega^2}$$

FIGURE 4. Adaptive Missile Autopilot Operation.

REFERENCES

1. W. W. Willman. "Effects of Strapdown Seeker Scale-Factor Uncertainty on Optimal Guidance," *J. Guid. Control and Dyn.*, Vol. 11 (1988), p. 199.

2. A. Gelb, Ed. *Applied Optimal Estimation*. Cambridge, Mass., MIT Press, 1974.

3. F. C. Schweppe. *Uncertain Dynamic Systems*. Englewood Cliffs, N.J., Prentice-Hall, 1973.

4. H. Raiffa and R. Schlaifer. *Applied Statistical Decision Theory*. Cambridge, Mass., MIT Press, 1972.

5. R. L. Stratonovich. "On the Theory of Optimal Control: Sufficient Coordinates," *Automat. Remote Control*, Vol. 23 (1963), p. 847.

6. S. E. Dreyfus. *Dynamic Programming and the Calculus of Variations*. New York, Academic Press, 1965.

7. W. W. Willman. "Some Formal Effects of Nonlinearities in Optimal Perturbation Control," *J. Guid. and Control*, Vol. 2 (1979), p. 99.

8. A. E. Bryson, Jr., and Y.-C. Ho. *Applied Optimal Control*. Washington, D.C., Hemisphere, 1975.

9. M. Tahk, M. M. Briggs, and P. K. A. Menon. "Applications of Plant Inversion Via State Feedback to Missile Autopilot Design," in *Proceedings of the 27th Conference on Decision and Control, December 7-9, 1988, Austin Texas*. New York, N.Y., IEEE, 1988, p. 730.